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Extension of Ferrari's Method to Solve Reducible Quintic Equation

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Abstract: Ferrari's method has been used to solve a biquadratic equation. In this paper the method has been extended to solve a reducible quintic equation. The general quintic equation can divided in reducible and irreducible quantities. In this paper the technique to solve reducible quintic equation is as follows. First multiply the quintic equation by variable 'x' the equation becomes a sextic equation (sixth degree equation). Then splitting the resulting equation into product of two cubic factors. The sextic equation will split into two seprate cubic equation , which can be solved by cardan,s method.

Introduction

In mathematics a quintic function is a function of the form $f(x)=a_0x^5+a_1x^4+a_2x^3+a_3x^2+a_4x+a_5$ where($a_0\neq0$) or in others words a function defined by a polynomial of degree 5, getting f(x)=0 produce a quintic equation of the form $a_0x^5+a_1x^4+a_2x^3+a_3x^2+a_4x+a_5=0$, where a_i's are rational. There are two type of quintic reducible and irreducible quantities, our main concern is about reducible quantities. A quintic is reducible in x if it reducible to (linear x quadratic) or (quadratic x cubic). . Otherwise it is said to be irreducible. Solving quintic equation [4,7,8] in term of redical was a major problem in algebra from 16th century, cubic and biquadratic equation [1,2,3] wheresolved until the half of the century, when the impossibility of such a general solution was proved (Abel-Ruffini theorem) some quintic equation [6,7] can be solved in term of radical. These include the reducible quantities and solvable irreducible quantities for characterizing solvable quantities . "Gveriste galois" developed technique which gave a rise to group theory and galois theory [5]. But reducible quantities are always solvable in radicals [7].

Explanation of method

every fifth degree equation $A_0x^5+A_1x^4+A_2x^3+A_3x^2+A_4x+A_5=0$ …(1).

(Where ($A_0\neq0$) and A_i's are rational) can be reduce to the form $x^5+a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0$,

$a_i\in Z$ …(2) (by multiplying the root of (1) by suitable factor)

If the quintic equation (1) is reducible over rational then (2) is also.

Then equation (2) will be reducible in the form either (linear x quartic) or (quadratic x cubic) there is no other form of reducible quantities.

Case I: if (2) =linear x quartic =0

➔ (2) has a linear factor ➔ equation (2) has an integer solution because coefficient of x^5 is one in (2) which can be find out easily using properties of continuous function i.e. if f(α)= +ve and f(β)= -ve. Then there exits c ϵ (α,β) s.t. f(c)=0 (if there does not exist such an integer) then (2) can not be reduce to the form Linear x quartic.

Case II: if the quintic equation (2) is reducible in the form (quadratic x cubic)

f(x)= (quadratic) x (cubic)=0

➔xf(x)=x((quadratic) x (cubic))=0

=(cubic) x (cubic)

➔xf(x)=0 can be reduces in two cubic factor

Now let us try to reduce xf(x) in two cubic factor

$xf(x)=0$ ➔ $x(x^5+a_0x^4+a_1x^3+a_2x^2+a_3x+a_4)=0$

➔ $x^6+a_0x^5+a_1x^4+a_2x^3+a_3x^2+a_4x=0$ …(3)

➔ $x^6+a_0x^5=-a_1x^4-a_2x^3-a_3x^2-a_4x$

Adding $(a_0^2x^4)/4$ both side

$x^6+a_0x^5+(a_0^2x^4)/4 =(a_0^2x^4)/4 -a_1x^4-a_2x^3-a_3x^2-a_4x$

➔ $(x^3+(a_0/2)x^2)^2=((a_0^2-4a_1)/4)x^4 -a_2x^3-a_3x^2-a_4x$ …(4)

Introducing $\lambda_1 x+\lambda_2$

$(x^3+\frac{a_0}{2}x^2+\lambda_1 x+\lambda_2)^2=(x^3+\frac{a_0}{2}x^2)^2+2(\lambda_1 x+\lambda_2)(x^3+\frac{a_0}{2}x^2)+(\lambda_1 x+\lambda_2)^2$ …(5)

Putting the value of $(x^3+\frac{a_0}{2}x^2)$ from (4) in (5)

$(x^3+\frac{a_0}{2}x^2+\lambda_1 x+\lambda_2)^2=\frac{a_0^2-4a_1}{4}x^4-a_2x^3-a_3x^2-a_4x+2(\lambda_1 x+\lambda_2)(x^3+\frac{a_0}{2}x^2)+(\lambda_1 x+\lambda_2)^2$

$=(2\lambda_1+\frac{a_0^2-4a_1}{4})x^4+(a_0\lambda_1+2\lambda_2-a_2)x^3+(a_0\lambda_2+\lambda_1^2-a_3)x^2+(2\lambda_1\lambda_2-a_4)x+\lambda_2^2$ …(6)

R.H.S. of (6) is of the form

$ax^4+bx^3+cx^2+dx+e$ …(7)

to mak R.H.S. a perfect square

if R.H.S. $=(Ax^2+Bx+C)^2$

➔ $ad^2=b^2e$ …(8)

$(b^2-4ac)^2=64a^3e$ …(9)

Comparing (6) and (7) we have

$a=2\lambda_1+\frac{a_0^2-4a_1}{4}$, $b=(a_0\lambda_1+2\lambda_2-a_2)$,

$c=a_0\lambda_2+\lambda_1^2-a_3$, $d=2\lambda_1\lambda_2-a_4$, $e=\lambda_2^2$

these putting in (8) and (9)

$(2\lambda_1 + \frac{a_0^2-4a_1}{4})(2\lambda_1\lambda_2 - a_4)^2 == (a_0\lambda_1 + 2\lambda_2 - a_2)^2(\lambda_2^2)$ ...(A)

$[(a_0\lambda_1 + 2\lambda_2 - a_2)^2 - 4(2\lambda_1 + \frac{a_0^2-4a_1}{4})(a_0\lambda_2 + \lambda_1^2 - a_3)]^2 = 64(2\lambda_1 + \frac{a_0^2-4a_1}{4})^3\lambda_2^2$ ...(B)

Let us take

$f_1(\lambda_1,\lambda_2)=0$ [➔A]

$f_2(\lambda_1,\lambda_2)=0$ [➔B]

now our main intension is to find common solution of (A) and (B)

if equation (3) is reducible in the form (cubic x cubic)

then (A) and (B) must have a rational common solution.

To find that common solution

Choose λ_1 such that $(2\lambda_1 + \frac{a_0^2-4a_1}{4})=\alpha^2$

Where α is rational

Find the value of λ_2 from (A) it must be rational. If these rational values of λ_1 and λ_2 also satisfy (B) then the equation (2) will be reducible over rational in the form (quadratic x cubic)

Now R.H.S. of (6) become a perfect square of the type $(Ax^2+Bx+C)^2$

Then by equation (6)

$(x^3 + \frac{a_0}{2}x^2 + \lambda_1 x + \lambda_2)^2 = \pm(Ax^2+Bx+C)$

By solving these two cubic equation we will get 6 root of the equation (3) leave the root x=0. The remaining 5 root are the root of quintic equation (2).

Example:

1. $2x^5 - 10x^3 + 12x - x^4 + 5x^2 - 6 = 0$

$2x^5 - x^4 - 10x^3 + 5x^2 + 12x - 6 = 0$ ...(1)

Multiplying the root of equation (1) by 2, y=2x

$2y^5-2y^4-40y^3+40y^2+192y-192=0$

➔$y^5-y^4-20y^3+20y^2+96y-96=0$ ...(2)

Clearly y=1 is a root of (2) so x=1/2 is a root of (1) (2x-1) is a factor of (1) other four root of (1) are given by $\pm\sqrt{2}, \pm\sqrt{3}$.

2.

$x^5 + x^4 - 2x^3 - 2x^2 - 2x + 1=0$ ...(1)

➔$x(x^5 + x^4 - 2x^3 - 2x^2 - 2x + 1)=0$

$\Rightarrow x^6 + x^5 - 2x^4 - 2x^3 - 2x^2 + x = 0$

$\Rightarrow x^6 + x^5 = 2x^4 + 2x^3 + 2x^2 - x$

Adding $\frac{1}{4}x^4$ both side

$$x^6 + x^5 + \frac{1}{4}x^4 = (\frac{1}{4} + 2)x^4 + 2x^3 + 2x^2 - x$$

$(x^3 + \frac{1}{2}x^2)^2 = \frac{9}{4}x^4 + 2x^3 + 2x^2 - x$ …(2)

Introducing $\lambda_1 x + \lambda_2$ we get

$(x^3 + \frac{1}{2}x^2 + \lambda_1 x + \lambda_2)^2 = (x^3 + \frac{1}{2}x^2)^2 + 2(\lambda_1 x + \lambda_2)(x^3 + \frac{1}{2}x^2) + (\lambda_1 x + \lambda_2)^2$

$= \frac{9}{4}x^4 + 2x^3 + 2x^2 - x + 2(\lambda_1 x + \lambda_2)(x^3 + \frac{1}{2}x^2) + (\lambda_1 x + \lambda_2)^2$

$= (2\lambda_1 + \frac{9}{4})x^4 + (2 + \lambda_1 + 2\lambda_2)x^3 + (2 + \lambda_2 + {\lambda_1}^2)x^2 + (-1 + 2\lambda_1\lambda_2)x + {\lambda_2}^2$ …(3)

Comparing with (7)

a=$(2\lambda_1 + \frac{9}{4})$, b=$(2 + \lambda_1 + 2\lambda_2)$, c=$(2 + \lambda_2 + {\lambda_1}^2)$,

d=$(-1 + 2\lambda_1\lambda_2)$, e=${\lambda_2}^2$

putting in (8) and (9)

$(2\lambda_1 + \frac{9}{4})\ (-1 + 2\lambda_1\lambda_2)^2 = (2 + \lambda_1 + 2\lambda_2)^2{\lambda_2}^2$ …(A)

$[(2 + \lambda_1 + 2\lambda_2)^2 - 4(2\lambda_1 + \frac{9}{4})\ (2 + \lambda_2 + {\lambda_1}^2)]^2 = 64(2\lambda_1 + \frac{9}{4})^2{\lambda_2}^2$ …(B)

$\lambda_1 = -1$ in (A)

$\frac{1}{4}(-1 - 2\lambda_2)^2 = (1 + 2\lambda_2)^2{\lambda_2}^2$

$\frac{1}{4}(1 + 2\lambda_2)^2 - (1 + 2\lambda_2)^2{\lambda_2}^2 = 0$

$(1 + 2\lambda_2)^2\left(\frac{1}{4} - \lambda_2^2\right) = 0$

$\Rightarrow 1 + 2\lambda_2 = 0$ and $\frac{1}{4} - \lambda_2^2 = 0$

$\Rightarrow \lambda_2 = -\frac{1}{2}$ and $\lambda_2 = \pm\frac{1}{2}$

⇨ $(-1 - \frac{1}{2}),\ (-1, \frac{1}{2})$

At $(-1 - \frac{1}{2})$

a=$\frac{1}{4}$, b=2-1-1=0, c=2-$\frac{1}{2} + 1 = \frac{5}{2}$, d=-1+1=0, e=$\frac{1}{4}$

$\Rightarrow (-4.\frac{1}{4}.\frac{5}{2}\ )^2 \neq 64(\frac{1}{4})^3.\frac{1}{4}$

At $(-1,\frac{1}{2})$

$[(2-1+1)^2-4.(\frac{1}{4})(2+\frac{1}{2}+1)]^2=64.(\frac{1}{4})^3\frac{1}{4}$

$[4-\frac{7}{2}]^2=\frac{1}{4}$

¼= ¼ then

Putting $(\lambda_1,\lambda_2)=(-1,\frac{1}{2})$

$(x^3+\frac{1}{2}x^2+\lambda_1 x+\lambda_2)^2=\frac{1}{4}x^4+2x^3+\frac{7}{2}x^2-2x+\frac{1}{4}$

➔ $(x^3+\frac{1}{2}x^2-x+\frac{1}{2})^2=(\frac{1}{2}x^2+2x-\frac{1}{2})^2$

➔ $(x^3+\frac{1}{2}x^2-x+\frac{1}{2})=\pm(\frac{1}{2}x^2+2x-\frac{1}{2})$

➔ $x^3+\frac{1}{2}x^2-x+\frac{1}{2}-\frac{1}{2}x^2-2x+\frac{1}{2}=0$ and $x^3+\frac{1}{2}x^2-x+\frac{1}{2}+\frac{1}{2}x^2+2x-\frac{1}{2}=0$

➔ $x^3-3x+1=0$ *and* $x^3+x^2+x=0$

➔ $x^3-3x+1=0$ *and* $x(x^2+x+1)=0$

➔ $x=2\cos\frac{2\pi}{9}, 2\cos\frac{8\pi}{9}, 2\cos\frac{14\pi}{9}$ and $x=0, \frac{-1\pm i\sqrt{3}}{2}$

So the root of equation (1) are $2\cos\frac{2\pi}{9}, 2\cos\frac{8\pi}{9}, 2\cos\frac{14\pi}{9}, \frac{-1\pm i\sqrt{3}}{2}$.

Conclusion

In this study, the given quintic is first converted to a Sextic equation by adding a root, and the resulting sextic equation is split into two cubic factor by Ferrari method .The resultant cubic equation are then solved to obtained five roots of the given quintic. So the proposed work of this paper has been done.

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